

# Maximum Entropy Charge-Constrained Run-Length Codes

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**Abstract**—This paper presents a study of runlength-limiting codes that have a null at “zero frequency” or “dc.” The class of codes or sequences considered is specified by three parameters:  $(d, k, c)$ . The first two constraints,  $d$  and  $k$ , put lower and upper bounds on the runlengths, while the “charge constraint,”  $c$ , is responsible for the spectral null. A description of the combined  $(d, k, c)$  constraints, in terms of a variable length graph, and its adjacency matrix,  $A(D)$ , are presented. This new approach to describing these constraints leads to a concise description of many properties of the charge-constrained codes. The maximum entropy description of the constraint described by a runlength graph is presented as well as the power spectral density. The results are used to study several examples of  $(d, k, c)$  constraints: the  $(d = 0, k = 1, c = 1)$  constraint (which is equivalent to the “FM” code), the  $(d = 1, k = 3, c = 3)$  constraint (which has capacity  $1/2$ ), and the  $(0 \leq d \leq 2, k = 3, c = 2)$  constraints. The procedures of finding the maxi-entropy distribution and power spectral density involve the determination of certain eigenvalues and eigenvectors of an associated with adjacency matrix  $A(D)$ . The eigenvalues and eigenvectors of the classes of  $(d, k = 2c - 1, c)$  and  $(d, k = d + 1, c)$  constraints for  $(c = 1, 2, \dots)$ , are shown to satisfy certain second-order recursive equations. These equations are solved using the theory of Tchebysheff polynomials. The results obtained are useful tools to compute the maximum entropy distributions and power spectral densities of many constraints.

## I. INTRODUCTION

THIS paper presents a study of runlength-limiting codes that have a null at “zero frequency” or “dc.” Such codes are described by sequences that satisfy a “running digital sum” or “charge” constraint. The class of codes or sequences considered are specified by three parameters  $(d, k, c)$ ; the  $d$  parameter controls the minimum runlength, the  $k$  parameter controls the maximum runlength, and the  $c$  parameter determines the charge constraint [1]–[6], [32].

The paper begins by describing constrained binary signals and codes. It presents a description of the  $(d, k, c)$  constraints in terms of a variable length graph [7], [8]. This new and (in our view) most natural approach to de-

scribing these constraints leads to a concise description of many properties of the charge-constrained codes.

The maximum entropy distribution of a code or constraint described by a variable length graph is presented; the results follow from the Perron–Frobenius Theory [9]–[13]. It is shown how the results of [14], together with the maximum entropy distribution of the  $(d, k, c)$  runlength graph, determine the power spectral density of these constraints [15]–[22]. The result is Theorem 1.

The results are used to study several examples of  $(d, k, c)$  constraints. First, the  $(d = 0, k = 1, c = 1)$  constraint, which is equivalent to the “FM” code, is used to show how our approach yields a well-known power spectral density result. Next, a new result, the maximum entropy distribution and power spectral density of the  $(d = 1, k = 3, c = 3)$  constraint, is obtained [2], [23]. This constraint is of interest because it has a positive  $d$  constraint and a capacity that is rational,  $1/2$  (a “rare” event [23]). The closed-form solution to the power spectral density shows that the answer is a rational function over the rational numbers  $Q$ , not the extension field  $Q[\sqrt{2}]$ , as expected by the theory. (This leads to a “conjecture” concerning the minimal field over which the power spectral density is defined.) Finally,  $(0 \leq d \leq 2, k = 3, c = 2)$  constraints are presented so that the effect of the  $d$  constraint can be seen and to demonstrate certain issues of the theory.

In the closing sections of the paper, it is described how the eigenvalues and eigenvectors of the classes of  $(d, k = 2c - 1, c)$  and  $(d, k = d + 1, c)$  constraints  $(c = 1, 2, \dots)$  satisfy certain second-order recursions. The solution of the recursive equations relies on the theory of Tchebysheff polynomials; the theory is briefly reviewed. The results obtained are useful tools for computing the maximum entropy distributions and power spectral densities of many constraints (for example, these constraints cover all the examples presented except for the “ $(d = 1, k = 3, c = 3)$ ” constraint).

## II. MAXIMUM ENTROPY, CONSTRAINED, BINARY SIGNALS

### A. Constrained Binary Signals

In a variety of data communications and storage systems [2], [5], [6], [18], information is represented by a synchronous, binary *transmit signal* of the form

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$$w(t) = \sum_{j=0}^{\infty} a_j B_{\Delta}(t - j\Delta), \quad a_j \in \{-1, +1\},$$

$$B_{\Delta}(t) = \begin{cases} 1 & 0 \leq t < \Delta \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

where the *coded sequence*  $\{a_j\}$  is the unique representation of the *message sequence*  $\{m_j\}$  ( $m_j \in \{-1, +1\}$ ). In practice, the encoding of the message is provided at a fixed rational rate  $R = k/n$ , and density  $D = R/\Delta$ . A practical encoder (or *encoding function*) is implemented as a stationary,  $k$  input,  $n$  output, *finite state machine* (FSM) with  $M$  states; a “block code” is a trivial FSM with  $M = 1$  state. Examples of block encoders are: 1) the rate  $R = 1$ , non-return to zero (NRZ),  $a_j = m_j$ ; and 2) the rate  $R = 1/2$ , Manchester Code,  $(a_{2j}, a_{2j+1}) = (-m_j, m_j)$ . Examples of encoders with nontrivial memory are: 3) the rate  $R = 1$ , non-return to zero inverted (NRZI) with two states,  $s_j = a_{j-1} \in \{-1, +1\}$ ,  $a_j = m_j s_j$ ; and 4) the rate  $R = 1/2$ , modified frequency modulation (MFM) code with four states,  $(s_j^1, s_j^2) = (m_{j-1}, a_{2j-1}) \in \{-1, +1\}^2$

$$(a_{2j}, a_{2j+1}) = \begin{cases} (+s_j^1, -s_j^2) & m_j = -1; \\ (-s_j^1, -s_j^2) & m_j = +1, s_j^2 = +1; \\ (+s_j^1, +s_j^2) & m_j = +1, s_j^2 = -1. \end{cases}$$

Encoding functions, such as those described above, constrain the set of possible transmitted signals,  $w(t)$ , that can appear at the output of an encoder. The set of output sequences,  $\{a_j\} \in \mathcal{C}$ , is called the *codeword set* or simply the *code*.

In this paper, we are interested in characterizing statistical properties of codes for the class of “ $(d, k, c)$ ” constraints (in the above examples: 1)  $(d = 0, k = \infty, c = \infty)$ ; 2)  $(d = 0, k = 1, c = 1)$ ; 3)  $(d = 0, k = \infty, c = \infty)$ ; 4)  $(d = 1, k = 3, c = \infty)$ ). The statistical aspects of a code,  $\mathcal{C}$ , can be introduced by assuming that input,  $\{m_j\}$ , to an encoder is a random sequence (i.e., an independent, identically distributed, Bernoulli  $(1/2)$ , binary sequence). This randomness at the encoder places a probability measure (or distribution) on the code  $\mathcal{C}$ ; it is this measure that determines the statistical properties and is the measure of interest in this paper. Note that the maximum entropy measure on the code,  $\mathcal{C}$ , makes the code sequence  $\{a_j\}$  and the transmit signal  $w(t)$  random processes. In this way, these processes have well-defined power spectral densities. As will be demonstrated, these densities can be determined algebraically.

In general, the distribution on the code  $\mathcal{C}$  can be determined as the solution to a maximum entropy problem. The solution is independent of the actual encoding function and can be characterized explicitly (see Section II-D). Thus, for example, as far as we are concerned, both NRZ and NRZI are the same (they both place the uniform measure on the set of all binary strings) even though they involve distinct mappings from the set of messages onto  $\mathcal{C}$ . This is also true for the Manchester code since other encoders for this code (Bi-phase, frequency modulation)

produce the same set of strings,  $\mathcal{C}$ , with the same probability measure.

A binary encoding function defines a code  $\mathcal{C}$  (the set of binary output strings) and a probability measure on the code. Another way to define a code  $\mathcal{C}$  is in terms of simple constraints or restrictions on the set of allowed binary sequences. While it is sometimes possible to generate a given code  $\mathcal{C}$ , described in terms of constraints, as the set of outputs of a FSM encoder, this is usually not the case. (A necessary and sufficient condition for this to occur is for the capacity of the code to be a rational number [8], [23], [24], see Section II-D). However, a natural notion of probability measure for the code  $\mathcal{C}$  is still defined in terms of a maximum entropy distribution [9], [17], [20]–[22]. Thus, statistical measures for codes described in terms of constraints are well-defined and meaningful. We might note that, in practice, encoders are designed to generate sequences that satisfy a simple constraint at a fixed rational rate that is “close” to the capacity of the constraint. In this way, the code generated by the encoder is a subset of the code defined by the constraint [7], [8], [24]. If the rate of the encoder is close to the capacity of the constraint, then one would expect that the statistical properties of the code described by the constraint “closely” approximate the statistics of the code generated by the encoder. It is for this reason that we study the statistics of the maximum entropy, runlength constraints.

### B. The $(d, k, c)$ Runlength Constraints

The  $(d, k, c)$  constraints are easily described in terms of runlengths (see Fig. 1). A runlength of a binary signal is equal to the length of time that the signal is constant. Define the *transition times* of the transmit signal  $w(t)$ :

$$t_j = \inf \{t | t > t_{j-1}, w(t) \neq w(t_{j-1}^-)\}$$

then the runlengths

$$T_1 = t_1 - t_0, T_2 = t_2 - t_1, T_3 = t_3 - t_2, \dots,$$

$$T_j = t_j - t_{j-1}, \dots$$

Note that since  $w(t)$  is synchronous (1), the runlengths are multiples of the clock period  $\Delta$ . It should be clear that the runlengths of the signal  $w(t)$  are determined by the runlengths  $N_j = T_j/\Delta$  of the coded binary sequence  $\{a_j\}$

$$n_j = \min \{i | i > n_{j-1}, a_i \neq a_{n_{j-1}}\}, \quad N_j = n_j - n_{j-1}.$$

The  $(d, k)$  runlength constraints maintain uniform lower and upper bounds on the runlengths, for all  $j$ ,

$$T_{\min} = \Delta(d + 1) \leq T_j \leq T_{\max} = \Delta(k + 1)$$

or, equivalently,

$$N_{\min} = (d + 1) \leq N_j \leq N_{\max} = (k + 1).$$

The upper bound,  $T_{\max}$ , is usually imposed for reasons of clock recovery while the lower bound,  $T_{\min}$ , is used to help the detection process by mitigating some of the effects of intersymbol interference [5].

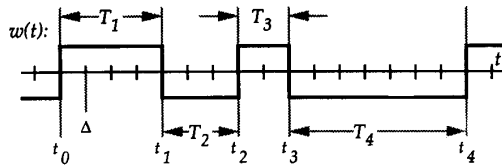


Fig. 1. The run-lengths of a binary signal  $w(t)$ .

The *charge constraint* ( $c$ ) is used to maintain a uniform bound on the magnitude of the integral (i.e., “running digital sum”) of the transmit signal, for all  $t$ ,

$$\frac{1}{\Delta} \left| \int_0^t w(\tau) d\tau \right| \leq c.$$

This constraint guarantees that the spectral component (both the discrete and continuous spectra) of  $w(t)$  at zero frequency is zero for all codewords [1], [3], [4]; this is important in a variety of applications [2], [5], [6]. In addition, it has been recently demonstrated that codes satisfying a charge constraint have “good” noise immunity properties when they are used over channels with transfer functions that have a spectral null at zero frequency [6].

It is interesting to note that the charge constraint is also described in terms of the runlengths of the codewords. First, we note that the charge constraint need only be satisfied at the transition times, for all  $j$ ,

$$\frac{1}{\Delta} \left| \int_0^{t_j} w(\tau) d\tau \right| \leq c$$

and at these times

$$\frac{1}{\Delta} \left| \int_0^{t_j} w(\tau) d\tau \right| = \left| \sum_{i=1}^j (-1)^i N_i \right| = |U_j|$$

where the sequence  $U_j = N_j - U_{j-1}$ ,  $U_0 = 0$ . Thus, the charge constraint is equivalent to  $|U_j| \leq c$  for all  $j$ .

The  $(d, k, c)$  constraint combines the runlength and charge constraints. A binary sequence satisfies  $(d, k, c)$  if and only if the runlengths  $\{N_j\}$  satisfy, for all  $j$ ,

$$d + 1 \leq N_j \leq k + 1 (\leq 2c) \tag{2a}$$

$$d + 1 - c \leq U_j \leq c. \tag{2b}$$

Note that the charge constraint imposes a maximum run-length since  $N_j = U_j + U_{j-1} \leq 2c$ ; thus  $N_{\max} \leq 2c$  or in terms of the “ $k$ ” constraint,  $k \leq 2c - 1$ . Also, since runlengths are bounded from below,  $N_j \geq d + 1$ , the  $\{U_j\}$  sequence can take on only  $2c - d$  values since  $U_j \leq c$  and  $U_j = N_j - U_{j-1} \geq d + 1 - c$ .

### C. The Runlength Graph

The code  $\mathcal{C}$  of virtually every practical encoder or run-length constraint (e.g., the  $(d, k, c)$  constraint) can be represented by an irreducible, finite state, runlength graph (see Fig. 2). A runlength graph,  $\mathcal{G} = (\mathcal{S}, \mathcal{E}, l)$ , consists of a finite set of  $|\mathcal{S}|$  states and a set of directed edges  $\mathcal{E} = \cup_{m,n \in \mathcal{S}} \mathcal{E}_{m,n}$ . Each edge  $e \in \mathcal{E}_{m,n}$  represents a link

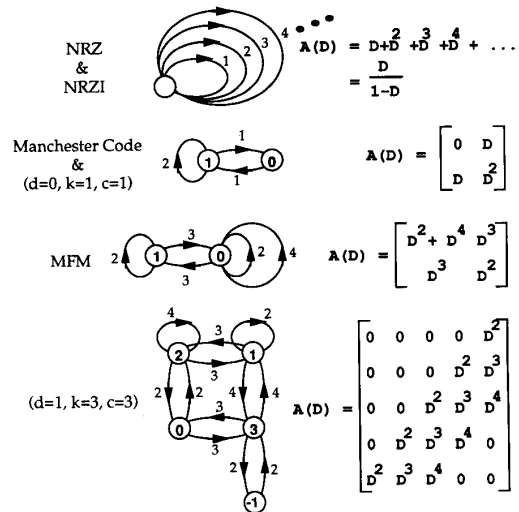


Fig. 2. Examples of run-length graphs and their adjacency matrices.

going from state  $m \in \mathcal{S}$  to state  $n \in \mathcal{S}$  and has an associated positive integer *length*,  $l(e)$ . The lengths distinguish the edges in the subsets  $\mathcal{E}_{m,n}$ : if  $e, e' \in \mathcal{E}_{m,n}$  then  $l(e) \neq l(e')$ . A runlength graph is a special instance of a “variable length graph” as defined in [8].

The runlength graph describes the set of possible sequences of runlengths of the code or constraint. A *path* through the graph,  $\gamma = e_1, e_2, e_3, \dots, e_L$ , is a valid sequence of edges; if  $e_{j-1} \in \mathcal{E}_{m,n}$ ,  $e_j \in \mathcal{E}_{n,i}$  then  $n = i$ . If every pair of states of the runlength graph is connected by at least one path, then the runlength graph is said to be *irreducible*. The length of a path,  $l(\gamma)$ , is the sum of the edge lengths,  $l(\gamma) = \sum_{j=1}^L l(e_j)$ . The paths through the graph represent valid runlength sequences determined by the lengths of the edges,  $N_1 = l(e_1)$ ,  $N_2 = l(e_2)$ ,  $N_3 = l(e_3)$ ,  $\dots$ .

It is assumed that the runlength graph is *lossless* or *finite to one*. This means that any doubly-infinite runlength sequence is produced by a finite number of valid edge sequences. Equivalently, this means that any finite edge path is uniquely determined from the initial state, the finite runlength string associated with the path, and the final state. One sufficient condition for this to happen is if the graph is *deterministic* or *unifilar*. In this case, for each state  $m$ , the lengths distinguish all the edges starting at state  $m$ ,  $\cup_n \mathcal{E}_{m,n}$  (or the graph is backward deterministic meaning the incoming edges,  $\cup_n \mathcal{E}_{n,m}$  are uniquely labeled). A weaker sufficient condition is that the graph is *right closing* meaning that, for each state  $m$ , the semi-infinite edge paths leaving state  $m$  are uniquely labeled (but not necessarily on the first step, as in the deterministic case). Similarly, a graph can be *left crossing* (backward right closing) to ensure that the graph is lossless. Furthermore, assuming the graph describing a constraint is lossless is not a restricting assumption; if a graph is not lossless, then the graph is not presenting the constraint in an efficient manner (i.e., it is a “bad” presentation of the

constraint). In fact, any constraint presented by a graph that is not lossless has another presentation, in terms of alternate graph, that is lossless.

Associated with the runlength graph of a code,  $\mathcal{C}$ , is a  $|\mathcal{C}| \times |\mathcal{C}|$ , square *adjacency matrix*,  $A(D)$ , where the  $m$ ,  $n \in \mathcal{C}^2$  term

$$A_{m,n}(D) = \sum_{e \in \mathcal{E}_{m,n}} D^{l(e)}.$$

The adjacency matrix is a matrix of polynomials (or rational functions) in the variable  $D$  over the integers,  $\mathbb{Z}$ . (Whenever the number of edges  $|\mathcal{C}|$  is finite, the adjacency matrix is a polynomial matrix; this is true whenever there is a "k" or "c" constraint.) The adjacency matrix gives a complete and compact algebraic representation of the runlength graph. As we shall see, this matrix is useful for determining the capacity and maximum entropy distribution of the code  $\mathcal{C}$  and for finding the power spectral densities of the associated random processes.

The general form of the adjacency matrix for the  $(d, k, c)$  constraints, derived from (2), has a regular structure. For these constraints, the matrix has size  $(2c - d) \times (2c - d)$  and is constant on the antidiagonals. If the  $k$  is non-trivial (i.e.,  $k + 1 < 2c$ ), the lower right of the matrix is zero,

$$A(D) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & D^{d+1} \\ 0 & 0 & 0 & \cdots & 0 & \cdots & D^{d+1} & D^{d+2} \\ \vdots & & & & & & & \vdots \\ 0 & D^{d+1} & D^{d+2} & \cdots & D^k & \cdots & & 0 \\ D^{d+1} & D^{d+2} & D^{d+3} & \cdots & D^{k+1} & \cdots & 0 & 0 \end{bmatrix}$$

while in the case of  $k = 2c - 1$ , the lower right corner is filled,

$$A(D) = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & D^{d+1} \\ 0 & 0 & 0 & \cdots & D^{d+1} & D^{d+2} \\ \vdots & & & & & \vdots \\ 0 & D^{d+1} & D^{d+2} & \cdots & & D^{2c-1} \\ D^{d+1} & D^{d+2} & D^{d+3} & \cdots & D^{2c-1} & D^{2c} \end{bmatrix}$$

Note that the  $(d, k, c)$  graph associated with (2) and the adjacency matrix is deterministic (and backward deterministic) and, thus, represents a lossless presentation for the constraint.

#### D. The Maximum Entropy Distribution

Given a code  $\mathcal{C}$ , one might ask for the probability distribution that maximizes the entropy  $H(a)$  of the code sequence  $\{a_j\}$ , subject to the constraint that the sequence is a member of the code  $\{a_j\} \in \mathcal{C}$  (this problem was originally stated and solved by Shannon [9]). When the code  $\mathcal{C}$  is described by a runlength graph  $\mathcal{G}$ , the solution to this problem is directly given in terms of the adjacency matrix of the graph  $A(D)$  [8], [9]. The maximizing distribution

is obtained by assigning the proper fixed probabilities to the edges of the graph,  $p^*(e)$ ,  $e \in \mathcal{E}$ . Such a fixed assignment makes the state sequence a stationary Markov chain with a *stationary distribution*  $\pi^*(m)$ ,  $m \in \mathcal{C}$ .

Consider a fixed probability assignment to the edges  $p(e)$ . Stochastically, the runlengths are then determined from the state sequence; for  $e \in \mathcal{E}_{m,n}$ ,  $m \in \mathcal{C}$ ,

$$\Pr(U_{j+1} = n, N_j = l(e) | U_j = m) = p(e),$$

$$\Pr(U_j = m) = \pi(m).$$

Note the *state transition probabilities* are obtained by summing the edge probabilities

$$\Pr(U_{j+1} = n | U_j = m) = \sum_{e \in \mathcal{E}_{m,n}} p(e).$$

In this way, the runlengths form a stationary random process that is a "random function" of a stationary Markov chain. These probabilities are conveniently described in matrix form [8]. Define the  $|\mathcal{C}| \times |\mathcal{C}|$ , square *edge probability matrix*,  $G(D)$ , where the  $m$ ,  $n$ th term

$$G_{m,n}(D) = \sum_{e \in \mathcal{E}_{m,n}} p(e) D^{l(e)}.$$

Then the *state transition probability matrix*  $P = G(D)|_{D=1}$  and the *stationary distribution vector* is the positive left eigenvector of eigenvalue 1, that sums to 1:  $\pi^t P = \pi^t$ ,  $\pi > 0$ ,  $\pi^t \mathbf{1} = 1$  (all vectors are taken to be column vectors).

The entropy of the runlength process is equal to the conditional entropy of the runlength, given the state

$$\begin{aligned} H(N) &\equiv \lim_{j \rightarrow \infty} \frac{1}{j} H(N_1, N_2, \dots, N_j) = H(N_1 | U_1) \\ &= - \sum_{m \in \mathcal{C}} \sum_{e \in \mathcal{E}_{m,n}} \pi(m) p(e) \log(p(e)) \end{aligned}$$

and is related to the entropy of the coded sequence by the relation  $H(a) = H(N)/E(N)$ , where  $E(N)$  is the expected length of a runlength. When the edge probabilities are judiciously chosen, the entropy of the coded sequence is maximized (note this maximizes the entropy of the coded sequence over **all** measures on the code  $\mathcal{C}$ ). The maximizing entropy of the coded sequence  $C \equiv H^*(a)$  is known as the *capacity* of the code; the number of strings of length  $n$  in  $\mathcal{C}$  grows exponentially for large  $n$ <sup>1</sup> as  $\sim 2^{nC}$ .

<sup>1</sup>To be specific,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log(|\mathcal{C}_n|) = C = H^*(a)$$

where  $\mathcal{C}_n$  is the set of  $a$  sequences of length  $n$  supported by the constraint.

The solution to the maximum entropy problem revolves around the Perron–Frobenius eigenvalue of the adjacency matrix  $A(D)$ . Define the *characteristic polynomial* of the adjacency matrix:

$$\chi_A(z) \equiv z^m \det(I - A(z^{-1})) = \chi_0 + \chi_1 z + \cdots + z^m$$

where  $I$  is the identity matrix and  $m$  is the degree of the determinate,  $m = \deg(\det(I - A(D)))$ . The *Perron–Frobenius eigenvalue* (PF-eigenvalue) is the largest magnitude of the (complex) roots of the characteristic polynomial

$$\lambda = \max_{\substack{\alpha \in \mathbb{C} \\ \chi_A(\alpha) = 0}} |\alpha|.$$

Note that for a variable length graph, the PF-eigenvalue is a “generalized eigenvalue”; if  $\chi_A(\alpha) = 0$ , then the matrix  $A(\alpha^{-1})$  has an eigenvalue of value 1 (i.e., there exists nontrivial solutions to  $A(\alpha^{-1})\mathbf{v} = \mathbf{v}$ ). From the *Perron–Frobenius Theory* [9]–[13], it is known that  $\lambda$  itself is a root of the characteristic polynomial  $\chi_A(\lambda) = 0$  and determines the capacity of the code  $C = \log(\lambda)^2$ .

Given the adjacency matrix  $A(D)$  and the PF-eigenvalue  $\lambda$ , the max-entropic probabilities are determined in the extension field of the rationals  $Q(\lambda)$ . The extension field  $Q(\lambda)$  is the smallest extension of the rationals  $Q$  that contain  $\lambda$ . Algebraically,  $Q(\lambda)$  is isomorphic to the ring of polynomials in  $z$  over  $Q$ , modulo the *minimal polynomial*  $m_\lambda(z)$  of  $\lambda$ ,  $Q(\lambda) = Q[z]/m_\lambda(z)$ . The minimal polynomial  $m_\lambda(z) \in \mathbb{Z}(z)$  is a monic integer polynomial of smallest degree for which  $\lambda$  is a root (i.e.,  $\lambda$  is an algebraic integer). The minimal polynomial  $m_\lambda(z)$  can be determined by factoring the characteristic polynomial  $\chi_A(z)$  over the rationals, finding the factor for which  $\lambda$  is a root, then normalizing (i.e., a monic integer polynomial).

To find the max-entropic probabilities, one must first find left and right, positive, eigenvectors of eigenvalue 1,

$$\begin{aligned} \mathbf{w}'A(\lambda^{-1}) &= \mathbf{w}', \quad A(\lambda^{-1})\mathbf{v} = \mathbf{v}, \quad \mathbf{w}, \mathbf{v} > 0, \\ \mathbf{w}, \mathbf{v} &\in \mathfrak{Q}(\lambda)^{|\mathfrak{E}|} \end{aligned}$$

(their existence is guaranteed by the Perron–Frobenius Theory). Then, the edge probability matrix

$$G(D) = V^{-1}A(\lambda^{-1}D)V$$

or

$$p^*(e) = \frac{v_n \lambda^{-l(e)}}{v_m}, \quad e \in \mathfrak{E}_{m,n}$$

and the stationary distribution vector

$$(\pi^*)' = \frac{1}{\mathbf{w}'\mathbf{v}} \mathbf{w}'V,$$

or

$$\pi^*(m) = \frac{w_m v_m}{\mathbf{w}'\mathbf{v}}, \quad m \in \mathfrak{E}$$

<sup>2</sup>It is at this point that the assumption that the graph is lossless is required. In general,  $C \leq \log(\lambda)$ , with equality in the lossless case (and thus the justification of the term “lossless”).

where  $V = \text{diag}(\mathbf{v})$  is obtained by forming a diagonal matrix from the right eigenvector  $\mathbf{v}$ .

### E. The Power Spectral Density

The max-entropic measure on the code  $\mathfrak{C}$  is one that makes the runlengths a stationary random process. Under this measure, the coded sequence  $\{a_j\}$  and the transmit signal  $w(t)$  are periodic random processes with period  $p \geq 1$  and  $p\Delta$ , respectively. The period  $p$  is a divisor of the period  $q$  of a runlength graph  $\mathfrak{G}$  for the code  $\mathfrak{C}$ . The *fundamental period*  $q$  of the runlength graph is the number equal to the greatest common divisor (GCD) of the lengths of the simple cycles of the runlength graph (a *cycle* is a path with a common beginning and ending state; a cycle is *simple* if no state, other than the beginning/ending state, is visited more than once). Again, the Perron–Frobenius Theory gives an algebraic characterization of the period. The period  $q$  is equal to the number of roots of the characteristic polynomial with magnitude  $\lambda$ . In fact,  $q$  is the largest integer such that  $(z^q - \lambda^q)$  divides  $\chi_A(z)$ . If  $q = 1$ , then the runlength graph is *aperiodic* or *stationary*.

The power spectral densities for periodic random processes are well defined by *phase randomization*. A stationary process is obtained by randomly shifting the periodic process. For example, if  $w(t)$  is periodic with period  $p\Delta$ , and  $p|q$ , then  $\tilde{w}(t) = w(t + \Theta)$  is a stationary process if  $\Theta$  is independent of  $w(t)$  and uniformly distributed over the interval  $[0, p\Delta]$  (or some integer multiple of the period, e.g.,  $[0, q\Delta]$ ). The power spectral density  $S_w(f)$  of  $w(t)$  is then defined to be the Fourier transform of the autocorrelation function of the stationary process  $\tilde{w}(t)$ .

In [14], it is shown how to obtain the power spectral densities  $S_x(e^{i2\pi f})$  and  $S_w(f)$  from the edge probability matrix  $G(D)$ . Define the sequence  $x_j = (a_j - a_{j-1})/2$ , then

$$S_x(e^{i2\pi f}) = \frac{1}{\sin(\pi f)^2} S_x(e^{i2\pi f}),$$

$$S_w(f) = \frac{1}{\Delta(\pi f)^2} S_x(e^{i2\pi f\Delta}) = \Delta \text{sinc}(f\Delta)^2 S_x(e^{i2\pi f\Delta})$$

and

$$\begin{aligned} S_x(D) &= \left( \frac{1}{\pi' G'(I) \mathbf{1}} \right) \pi' [(I + G(D))^{-1} \\ &\quad + (I + G(D^{-1}))^{-1} - I] \mathbf{1} \end{aligned}$$

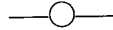
where  $I$  is the identity matrix and  $G'(I) = (d/dD)G(D)|_{D=1}$  is the derivative of edge probability matrix  $G(D)$  evaluated at unity. In our case, the edge probability matrix  $G(D)$  is obtained from the adjacency matrix  $A(D)$ , the PF-eigenvalue  $\lambda$ , and the right and left eigenvectors  $\mathbf{v}, \mathbf{w}$ . Thus, the following theorem holds.

*Theorem 1:* The spectrum is given by the formula

$$\begin{aligned} S_x(D) &= \left( \frac{1}{\lambda^{-1} \mathbf{w}' A'(\lambda^{-1}) \mathbf{v}} \right) \mathbf{w}' [(I + A(\lambda^{-1}D))^{-1} \\ &\quad + (I + A(\lambda^{-1}D^{-1}))^{-1} - I] \mathbf{v} \end{aligned}$$

where

$$A'(\lambda^{-1}) = \frac{d}{dD} A(D)|_{D=\lambda^{-1}}$$



Note the power spectral densities are obtained in terms of rational functions in  $D$  over the extension field  $\mathfrak{Q}(\lambda)$ .

### III. EXAMPLES OF $(d, k, c)$ RUNLENGTH CONSTRAINTS

In this section, we present several examples of  $(d, k, c)$  runlength constraints. First, the  $(d = 0, k = 1, c = 1)$  constraint, which is equivalent to the ‘‘FM’’ code, is used to show how our approach yields a well-known power spectral density result. Next, a new result, obtains the maximum entropy distribution and power spectral density of the  $(d = 1, k = 3, c = 3)$  constraint [2], [23]. This constraint is of interest because it has a positive  $d$  constraint and a capacity that is rational  $(1/2)$ . Finally,  $(0 \leq d \leq 2, k = 3, c = 2)$  constraints are presented so that the effect of the  $d$  constraint can be seen and to demonstrate certain issues of the theory.

#### B. The $(d = 1, k = 3, c = 3)$ Runlength Constraint

$$A(D) = \begin{bmatrix} 0 & 0 & 0 & 0 & D^2 \\ 0 & 0 & 0 & D^2 & D^3 \\ 0 & 0 & D^2 & D^3 & D^4 \\ 0 & D^2 & D^3 & D^4 & 0 \\ D^2 & D^3 & D^4 & 0 & 0 \end{bmatrix}$$

$$m_\lambda(z) = z^2 - 2$$

$$v = w = [2, 3\lambda, 6, 4\lambda, 4]^t$$

$$G(D) = \begin{bmatrix} 0 & 0 & 0 & 0 & D^2 \\ 0 & 0 & 0 & \frac{2D^2}{3} & \frac{D^3}{3} \\ 0 & 0 & \frac{D^2}{2} & \frac{D^3}{3} & \frac{D^4}{6} \\ 0 & \frac{3D^2}{8} & \frac{3D^3}{8} & \frac{D^4}{4} & 0 \\ \frac{D^2}{4} & \frac{3D^3}{8} & \frac{3D^4}{8} & 0 & 0 \end{bmatrix}$$

$$S_r(D) = \frac{(-1 + D)^2(-1 + D^{-1})^2 \begin{pmatrix} 39783 + 31768(D + D^{-1}) \\ + 19209(D^2 + D^{-2}) + 7276(D^3 + D^{-3}) \\ + 1263(D^4 + D^{-4}) + 1560(D^5 + D^{-5}) \\ + 1678(D^6 + D^{-6}) + 1840(D^7 + D^{-7}) \\ + 716(D^8 + D^{-8}) - 32(D^9 + D^{-9}) \\ - 104(D^{10} + D^{-10}) \end{pmatrix}}{72 \begin{pmatrix} 1505 + 528(D^2 + D^{-2}) - 412(D^4 + D^{-4}) - 436(D^6 + D^{-6}) \\ - 120(D^8 + D^{-8}) - 16(D^{10} + D^{-10}) - 32(D^{12} + D^{-12}) \end{pmatrix}}$$

$$S_w(f) = \Delta \operatorname{sinc} [f\Delta]^2 \frac{\begin{pmatrix} 8015 + 4544 \cos(2\pi f\Delta) - 626 \cos(4\pi f\Delta) \\ - 5920 \cos(6\pi f\Delta) - 6310 \cos(8\pi f\Delta) + 179 \cos(10\pi f\Delta) \\ - 44 \cos(12\pi f\Delta) + 1286 \cos(14\pi f\Delta) - 376 \cos(16\pi f\Delta) \\ - 676 \cos(18\pi f\Delta) - 176 \cos(20\pi f\Delta) + 104 \cos(22\pi f\Delta) \end{pmatrix}}{36 \begin{pmatrix} 1505 + 1056 \cos(4\pi f\Delta) - 824 \cos(8\pi f\Delta) - 872 \cos(12\pi f\Delta) \\ - 240 \cos(16\pi f\Delta) - 32 \cos(20\pi f\Delta) - 64 \cos(24\pi f\Delta) \end{pmatrix}}$$

#### A. The $(d = 0, k = 1, c = 1)$ Runlength Constraint

$$A(D) = \begin{bmatrix} 0 & D \\ D & D^2 \end{bmatrix}$$

$$m_\lambda(z) = z^2 - 2$$

$$v = w = [1, \lambda]^t$$

$$G(D) = \begin{bmatrix} 0 & D \\ \frac{D}{2} & \frac{D^2}{2} \end{bmatrix}$$

$$S_x(D) = \frac{6 - 4(D + D^{-1}) + (D^2 + D^{-2})}{8}$$

$$S_w(f) = \Delta \operatorname{sinc} [f\Delta]^2 (1 - \cos(2\pi f\Delta)).$$

This constraint is the basis of a family of codes, known as *Zero Modulation* (or ‘‘ZM’’), introduced by Patel in [2]. A study of the power spectral density of ZM was reported by Lindholm in [30].

The closed-form solution to the power spectral density shows that the answer is a rational function over the rational numbers  $\mathfrak{Q}$ , not the extension field  $\mathfrak{Q}[\sqrt{2}]$ , as expected by the theory.

This leads to the following:

**conjecture:** *The power spectral density of a  $(d, k, c)$  constraint is a rational function over the extension field  $\mathfrak{Q}[\lambda^2] \subset \mathfrak{Q}[\lambda]$ .*

Note that the conjecture holds true for every example in the paper (and every example we have tried).

We believe this result follows from the symmetry and

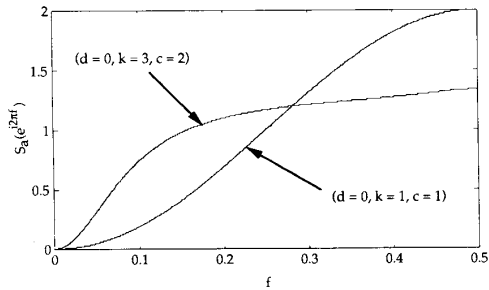


Fig. 3. Periodic spectrum,  $S_a(e^{i2\pi f})$  for  $d = 0$ .

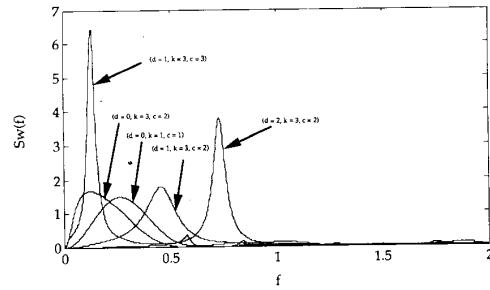


Fig. 5. Spectrum,  $S_w(e^{i2\pi f})$  at equal density ( $\Delta = \log_2(\lambda)$ ).

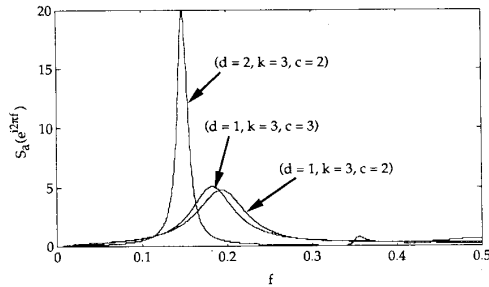


Fig. 4. Periodic spectrum,  $S_a(e^{i2\pi f})$  for  $d = 1, 2$ .

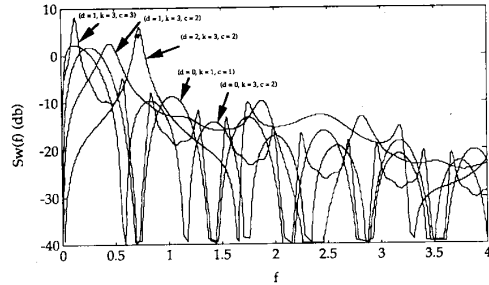


Fig. 6. Spectrum,  $S_w(e^{i2\pi f})$  at equal density ( $\Delta = \log_2(\lambda)$ ) in dB.

nonnegative constraints of power spectral densities. For example, as is well known, the power spectral density can be factored [31]  $S_x(D) = Kf(D)f(D^{-1})$ , where  $f(D)$  is a rational function over an extension of the rationals  $\mathfrak{D}[\gamma]$ . Thus, the power spectral density has a well-defined "square root." If one shows that the factor  $f(D)$  has coefficients in the field  $\mathfrak{D}[\lambda]$ , then it should be possible to prove the conjecture.

C. The  $(d = 0, k = 3, c = 2)$  Runlength Constraint

$$A(D) = \begin{bmatrix} 0 & 0 & 0 & D \\ 0 & 0 & D & D^2 \\ 0 & D & D^2 & D^3 \\ D & D^2 & D^3 & D^4 \end{bmatrix}$$

D. The  $(d = 1, k = 3, c = 2)$  Runlength Constraint

$$A(D) = \begin{bmatrix} 0 & 0 & D^2 \\ 0 & D^2 & D^3 \\ D^2 & D^3 & D^4 \end{bmatrix}$$

$$m_\lambda(z) = z^6 - z^4 - 2z^2 + 1$$

$$v = w = [-4 + 5\lambda^2 + 9\lambda^4, -6\lambda + 9\lambda^3 + 10\lambda^5, -9 + 14\lambda^2 + 14\lambda^4]^t$$

$$m_\lambda(z) = z^2 - 3$$

$$v = w = [1, \lambda, 2, \lambda]^t$$

$$G(D) = \begin{bmatrix} 0 & 0 & 0 & D \\ 0 & 0 & \frac{2D}{3} & \frac{D^2}{3} \\ 0 & \frac{D}{2} & \frac{D^2}{3} & \frac{D^3}{3} \\ \frac{D}{3} & \frac{D^2}{3} & \frac{2D^3}{9} & \frac{D^4}{9} \end{bmatrix}$$

$$S_x(D) = \frac{(-1 + D)^2(-1 + D^{-1})^2(7 + 3(D + D^{-1}))}{3(3D - D^{-1})(-D + 3D^{-1})}$$

$$S_w(f) = \Delta \text{sinc} [f\Delta]^2 \frac{4(4 - \cos(2\pi f\Delta) - 3 \cos(4\pi f\Delta))}{3(5 - 3 \cos(4\pi f\Delta))}$$

$$G(D) = \begin{bmatrix} 0 & 0 & D^2 \\ 0 & \frac{D^2}{\lambda^2} & \frac{D^3(-9 + 14\lambda^2 + 14\lambda^4)}{(-19 + 28\lambda^2 + 33\lambda^4)} \\ \frac{D^2(-4 + 5\lambda^2 + 9\lambda^4)}{(-14 + 19\lambda^2 + 28\lambda^4)} & \frac{D^3(-6 + 9\lambda^2 + 10\lambda^4)}{(-14 + 19\lambda^2 + 28\lambda^4)} & \frac{D^4}{\lambda^4} \end{bmatrix}$$

$$S_x(D) = \frac{\left( \begin{aligned} &(-782 + 1110\lambda^2 + 1454\lambda^4) + (236 - 338\lambda^2 - 432\lambda^4)(D + D^{-1}) \\ &+ (-18 + 40\lambda^2 + \lambda^4)(D^2 + D^{-2}) + (262 - 384\lambda^2 - 460\lambda^4)(D^3 + D^{-3}) \\ &+ (33 - 51\lambda^2 - 52\lambda^4)(D^4 + D^{-4}) + (-122 + 178\lambda^2 + 216\lambda^4)(D^5 + D^{-5}) \end{aligned} \right)}{\left( \begin{aligned} &(-3884 + 5580\lambda^2 + 7072\lambda^4) + (-1632 + 2372\lambda^2 + 2910\lambda^4)(D^2 + D^{-2}) \\ &+ (272 - 386\lambda^2 - 506\lambda^4)(D^4 + D^{-4}) + (506 - 740\lambda^2 - 892\lambda^4)(D^6 + D^{-6}) \end{aligned} \right)}$$

$$S_w(f) = \Delta \operatorname{sinc} [f\Delta]^2 \frac{\begin{aligned} &(-1016 + 1448\lambda^2 + 1876\lambda^4) + (-468 + 676\lambda^2 + 844\lambda^4) \cos(2\pi f\Delta) \\ &+ (152 - 192\lambda^2 - 336\lambda^4) \cos(4\pi f\Delta) \\ &+ (844 - 1220\lambda^2 - 1520\lambda^4) \cos(6\pi f\Delta) \\ &+ (488 - 712\lambda^2 - 864\lambda^4) \cos(8\pi f\Delta) \end{aligned}}{\left( \begin{aligned} &(-1942 + 2790\lambda^2 + 3536\lambda^4) + (-1632 + 2372\lambda^2 + 2910\lambda^4) \cos(4\pi f\Delta) \\ &+ (272 - 386\lambda^2 - 506\lambda^4) \cos(8\pi f\Delta) + (506 - 740\lambda^2 - 892\lambda^4) \cos(12\pi f\Delta) \end{aligned} \right)}$$

#### E. The ( $d = 2, k = 3, c = 2$ ) Runlength Constraint

$$A(D) = \begin{bmatrix} 0 & D^3 \\ D^3 & D^4 \end{bmatrix}$$

$$m_\lambda(z) = z^6 - z^2 - 1$$

$$v = w = [1 + 2\lambda^2 + \lambda^4, \lambda + 2\lambda^3 + 2\lambda^5]$$

$$G(D) = \begin{bmatrix} 0 & D^3 \\ \frac{D^3(1 + \lambda^2)^2}{2 + 4\lambda^2 + 3\lambda^4} & \frac{D^4}{\lambda^4} \end{bmatrix}$$

$$S_x(D) = \frac{\left( \begin{aligned} &(-12 - 20\lambda^2 - 16\lambda^4) + (6 + 10\lambda^2 + 8\lambda^4)(D + D^{-1}) \\ &+ (-3 - 5\lambda^2 - 4\lambda^4)(D^2 + D^{-2}) + (6 + 10\lambda^2 + 8\lambda^4)(D^3 + D^{-3}) \\ &+ (-3 - 5\lambda^2 - 4\lambda^4)(D^4 + D^{-4}) \end{aligned} \right)}{\left( \begin{aligned} &(-88 - 156\lambda^2 - 116\lambda^4) + (14 + 24\lambda^2 + 20\lambda^4)(D^2 + D^{-2}) \\ &- (34 + 58\lambda^2 + 44\lambda^4)(D^4 + D^{-4}) + (24 + 44\lambda^2 + 34\lambda^4)(D^6 + D^{-6}) \end{aligned} \right)}$$

$$S_w(f) = \Delta \operatorname{sinc} [f\Delta]^2 \frac{6 + 10\lambda^2 + 8\lambda^4 - (6 + 10\lambda^2 + 8\lambda^4) \cos(6\pi f\Delta)}{\left( \begin{aligned} &22 + 39\lambda^2 + 29\lambda^4 - (7 + 12\lambda^2 + 10\lambda^4) \cos(4\pi f\Delta) \\ &+ (17 + 29\lambda^2 + 22\lambda^4) \cos(8\pi f\Delta) - (12 + 22\lambda^2 + 17\lambda^4) \cos(12\pi f\Delta) \end{aligned} \right)}$$

#### IV. CLASSES OF RUNLENGTH CONSTRAINTS: $k = 2c - 1$ AND $k = d + 1$

For the cases  $k = 2c - 1$  and  $k = d + 1$  ( $c = 1, 2, \dots$ ), the eigenvalues and eigenvectors of  $A(D)$  are determined by second-order recursive relations. These recursions can be solved in terms of Tchebysheff polynomials. It is shown how to apply the theory of Tchebysheff polynomials to these problems.

#### A. Tchebysheff Polynomials

Tchebysheff polynomials are usually defined as polynomials in  $x$ ; these arise from the recursive relation

$$\Phi_{n+1}(x) = 2x\Phi_n(x) - \Phi_{n-1}(x) \quad (4.1)$$

where  $\Phi_0 \equiv 1$ ,  $\Phi_{-1} + \Phi_1 = 2x$  (note that either  $\Phi_1$  or  $\Phi_{-1}$  must also be specified to "solve" the recursion).



Given a similar recursion of the form

$$\Psi_{n+1} = E\Psi_n - F^2\Psi_{n-1} \quad (4.2a)$$

then it is easy to show that (4.1) is satisfied by identifying

$$\Phi_n = \frac{\Psi_n}{F^n\Psi_0} \quad \text{and} \quad 2x = \frac{E}{F} \quad (4.2b)$$

(assuming  $F, \Psi_0 \neq 0$ ).

Luke [25] defines two types of Tchebysheff polynomials: the polynomials (in  $x$ ),  $T_n(x)$  and  $U_n(x)$ , are known as Tchebysheff polynomials of the first and second kind, respectively. They are defined with (4.1) and the initial conditions

$$T_1(x) = x, \quad U_1(x) = 2x.$$

The Tchebysheff polynomials are concisely expressed in terms of trigonometric functions

$$T_n(x) = \cos(n \arccos(x))$$

$$U_n(x) = \frac{\sin((n+1) \arccos(x))}{\sin(\arccos(x))}.$$

The utility of the Tchebysheff polynomials is that any linear combination of  $T_n(x)$  and  $U_n(x)$ ,

$$\Phi_n = \alpha_1 T_n(x) + \alpha_2 U_n(x) \quad (4.3)$$

will solve a recursion of the form of (4.1) and conversely any solution can be put in this form. The coefficients  $\alpha_1$  and  $\alpha_2$  are solved from the initial conditions (recall  $\Phi_0 = 1, \Phi_{-1} + \Phi_1 = 2x$ ):

$$\alpha_1 = \frac{\Phi_{-1}}{x} = \frac{2x - \Phi_1}{x}$$

and

$$\alpha_2 = \frac{x - \Phi_{-1}}{x} = \frac{(\Phi_1 - x)}{x} = \frac{(\Phi_1 - \Phi_{-1})}{2x}.$$

Let  $\gamma \equiv \arccos(x)$ , then (4.3) can be written:

$$\Phi_n = \frac{\Phi_{-1} \cos(n\gamma)}{x} + \frac{(\Phi_1 - \Phi_{-1}) \sin((n+1)\gamma)}{\sin(\gamma)2x}. \quad (4.4a)$$

There are alternative formulas for  $\Phi_n$ . From the CRC tables [26], (4.4a) becomes:

$$\Phi_n = \frac{1}{2x \sin(\gamma)} [\Phi_1(\sin((n+1)\gamma)) - \Phi_{-1} \sin((n-1)\gamma)]. \quad (4.4b)$$

Now note that  $2 \cos(\gamma) = 2x$  and  $\Phi_0 = 1$  implies  $\Phi_1 = 2x - \Phi_{-1} = 2 \cos(\gamma) - \Phi_{-1}$ , and from (4.4b)

$$\Phi_n = \frac{1}{\sin(\gamma)} [(\sin((n+1)\gamma)) - \Phi_{-1} \sin(n\gamma)]. \quad (4.4c)$$

Also,  $\Phi_{-1} = 2 \cos(\gamma) - \Phi_1$  and from (4.4b)

$$\Phi_n = \frac{1}{\sin(\gamma)} [\Phi_1 \sin(n\gamma) - \sin((n-1)\gamma)]. \quad (4.4d)$$

From (4.2), if  $\Phi_n = 0$  then  $\Psi_n = 0$  (assuming  $\Psi_0 \neq 0$ ). Setting (4.4b), (4.4c) and (4.4d) equal to zero with  $\gamma = \arccos(x)$  shows:

$$\begin{aligned} \Phi_n = 0 &\Leftrightarrow \Phi_{-1} \sin((n-1) \arccos(x)) \\ &= \Phi_1 \sin((n+1) \arccos(x)) \\ &\Leftrightarrow \Phi_{-1} \sin(n \arccos(x)) \\ &= \sin((n+1) \arccos(x)) \\ &\Leftrightarrow \Phi_1 \sin(n \arccos(x)) \\ &= \sin((n-1) \arccos(x)). \end{aligned} \quad (4.5)$$

Thus given a recursion in the form of (4.1), knowledge of  $2x, \Phi_1$  (or  $\Phi_{-1}$ ) will directly give the solution of  $\Phi_n$  (4.4) and its roots (4.5).

### B. The $(d, k = 2c - 1, c)$ Constraint

The  $(d, k, c)$  constraint reduces to a  $(d, k = 2c - 1, c)$  constraint in the absence of a specific  $k$  constraint. Given a  $c$  constraint, the maximum run has length  $2c$  and so  $k \leq 2c - 1$ .

*Theorem 2:* Fix  $d \geq 0$  and let  $A(D)$  be the  $2c - d \times 2c - d$  adjacency matrix of the  $(d, k = 2c - 1, c)$  constraint. Define  $\Psi_n \equiv \det(I - A(D))$ , where  $n = c - \lfloor d/2 \rfloor$ . Then:

$$\Psi_{n+1} = (1 + D^2 - D^{2d+2})\Psi_n - D^2\Psi_{n-1}$$

where  $\Psi_0 = 1$  and

$$\Psi_{-1} = \begin{cases} 1 + D^d & d - \text{even} \\ 1 + D^{d-1} - D^{2d} & d - \text{odd} \end{cases}$$

Note:

$$\Psi_1 = \begin{cases} 1 - D^{d+2} - D^{2d+2} & d - \text{even} \\ 1 - D^{d+1} & d - \text{odd} \end{cases}$$

The PF-eigenvalue,  $\lambda$ , is the largest real valued solution to the equation

$$D^{-1} + D^{-d-1} = \frac{\sin[(c - d/2 + 1)\gamma(D^{-1})]}{\sin[(c - d/2)\gamma(D^{-1})]}$$

where

$$\gamma(D) \equiv \arccos\left(\frac{D + D^{-1} - D^{-2d-1}}{2}\right).$$

*Corollary 3:* In the case of a pure charge constraint ( $d = 0, k \geq 2c - 1$ ), Theorem 2 reduces to the previously known expression given in Chien [1]. In the case of  $d = 0$ ,

$$\gamma(D) = \arccos(D/2).$$

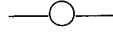
Thus, the PF-eigenvalue satisfies

$$\cos((c+1)\gamma(D^{-1})) \sin(\gamma(D^{-1})) = 0.$$

The largest value of  $D^{-1}$  that satisfies this equation is:

$$\lambda = 2 \cos \left( \frac{\pi}{2(c+1)} \right)$$

which is the same formula as presented in Chien.



*The Smith Canonical Form [10, 27]:* Fix  $d \geq 0$  and let  $A(D)$  be the adjacency matrix of the  $(d, k = 2c - 1, c)$  constraint, a matrix over the ring of polynomial over the reals. Then there exists unimodular  $2c - d \times 2c - d$  polynomial matrices  $E(D)$  and  $J(D)$  such that  $A(D) = E(D)B(D)J(D)$ . (A polynomial matrix is unimodular if it is a unit in the ring of matrices, i.e., in this case the determinant is a (nonzero) real constant.) The matrix  $B(D)$  is a diagonal matrix with diagonal elements  $b_1(D), b_2(D), \dots, b_n(D)$ , such that  $b_k(D)$  divides  $b_{k+1}(D)$  for  $1 \leq k < 2c - d$ . The matrix  $B(D)$  is known as the *Smith Canonical Form* of  $A(D)$ .

*Lemma 4:* The Smith form of  $A(D)$  is

$$B(D) = D^{d+1}I$$

where  $I$  is the  $2c - d \times 2c - d$  identity. The matrix  $E(D)$  is:

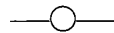
$$E(D) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ D & 1 & 0 & \cdots & 0 \\ D^2 & D & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ D^{2c-d-1} & \cdots & D^2 & D & 1 \end{pmatrix}$$

and the matrix  $J(D)$  is the matrix with all 1's on the anti-diagonal and zeros elsewhere

$$J(D) = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & & & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

Notice that the inverses  $J^{-1}(D) = J(D)$  and

$$E^{-1}(D) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ -D & 1 & 0 & \cdots & 0 \\ 0 & -D & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \vdots & 0 & -D & 1 \end{pmatrix}$$



The Smith Canonical Form is a useful tool for finding the eigenvector for  $A(\lambda^{-1})$ ; the result is stated in the following theorem.

*Theorem 5:* The left eigenvector,  $v(A(\lambda^{-1})v = v)$ , satisfies the recursion

$$v_{j+1} = (\lambda + \lambda^{-1} - \lambda^{-2d-1})v_j - v_{j-1}$$

which is solved by

$$v_i = \sin(i\gamma(\lambda))$$

where (as in Theorem 2)

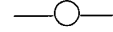
$$\gamma(D) \equiv \arccos \left( \frac{D + D^{-1} - D^{-2d-1}}{2} \right).$$

The edge probability matrix  $G(D)$  is given by:

$$G_{m,n}(D) = \begin{cases} \lambda^{2c-2d-m-n} \frac{\sin(n\gamma(\lambda))}{\sin(m\gamma(\lambda))} \cdot D^{m+n-2(c-d)} & 1 \leq m \leq 2c-d, 2c-d+1 \\ & \leq m+n \leq 4c-2d; \\ 0 & \text{otherwise} \end{cases}$$

and the stationary probability vector is given by

$$\pi_m^* = \frac{\sin^2(m\gamma(\lambda))}{\sum_{j=1}^{2c-d} \sin^2(j\gamma(\lambda))}, \quad 1 \leq m \leq 2c-d.$$



### C. The $(d, d+1, c)$ Constraint

In this section, the tightest nontrivial  $k$  constraint is analyzed. For a given value of  $d$ , let  $k = d + 1$ ; the run-lengths are constrained to be either  $d + 1$  or  $d + 2$ . This constraint has solutions of the form of Section III, but the derivation of the eigenvector is simpler. The adjacency matrix  $A(D)$  is the  $2c - d \times 2c - d$  matrix with  $D^{d+1}$  on the principal anti-diagonal and  $D^{d+2}$  on the first sub-anti-diagonal.

*Theorem 6:* Fix  $d \geq 0$  and let  $A(D)$  be the  $2c - d \times 2c - d$  adjacency matrix of the  $(d, k = d + 1, c)$  constraint. Define  $\Psi_n \equiv \det(I - A(D))$ , where  $n = c - \lfloor d/2 \rfloor$ . Then:

$$\Psi_{n+1} = (1 - D^{2d+2} - D^{2d+4})\Psi_n - D^{4d+6}\Psi_{n-1}$$

where

$$\Psi_0 = \begin{cases} 1 & d - \text{even} \\ D^{-d-1} & d - \text{odd} \end{cases}$$

and

$$\Psi_1 = \begin{cases} 1 - D^{d+2} - D^{2d+2} & d - \text{even} \\ 1 - D^{d+1} & d - \text{odd} \end{cases}$$

The PF-eigenvalue,  $\lambda$ , is the largest real valued solution to the equation

$$D^{d+1} - D^{-1} = \frac{\sin[(c - d/2 + 1)\gamma(D^{-1})]}{\sin[(c - d/2)\gamma(D^{-1})]}$$

where

$$\gamma(D) \equiv \arccos \left( \frac{D^{-2d-3} - D^{-1} - D}{2} \right).$$

**Theorem 7:** The left eigenvector,  $\mathbf{v}(A(\lambda^{-1})\mathbf{v} = \mathbf{v})$ , satisfies the recursion

$$v_{2c-d-j-1} = (\lambda^{2d+3} - \lambda^{-1}\lambda^{-1})v_{2c-d-j} - v_{2c-d-j+1}$$

which is solved by

$$v_i = \sin((2c - d - i + 1)\gamma(\lambda^{-1}))$$

where (as in Theorem 6)

$$\gamma(D) \equiv \arccos \left( \frac{D^{-2d-3} - D^{-1} - D}{2} \right).$$

The edge probability matrix  $G(D)$  is given by:

$$G_{m,n}(D) = \begin{cases} \lambda^{2c-2d-m-n} & \\ \frac{\sin((2c-d-n+1)\gamma(\lambda^{-1}))}{\sin((2c-d-m+1)\gamma(\lambda^{-1}))} & \\ \cdot D^{m+n-2(c-d)} & \\ 1 \leq m \leq 2c-d, d+1 & \\ \leq m+n \leq 4c-2d; & \\ 0 & \text{otherwise} \end{cases}$$

and the stationary probability vector is given by

$$\pi_m^* = \frac{\sin^2((2c-d-m+1)\gamma(\lambda^{-1}))}{2c-d} \cdot \frac{1}{\sum_{j=1}^{2c-d} \sin^2((2c-d-j+1)\gamma(\lambda^{-1}))},$$

$$1 \leq m \leq 2c-d.$$

The proofs of the Theorems follow from the presented theory of Tchebysheff polynomials and some delicate algebraic and trigonometric manipulation. Since the results themselves are (in our opinion) more interesting than the proofs, we omit them here. We also mention that, in addition to obtaining recursive relations for the characteristic polynomial and the components of the eigenvector, other basic facts are known. For example, a closed-form expression for  $(I - A(D))^{-1}$  has been developed. Anyone interested in the details is encouraged to obtain the theses [28], [29], in which these additional relations and complete proofs can be found.

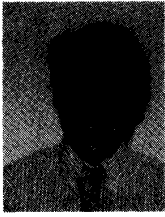
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